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Solitons in R_3 with winding number n

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Abstract. We present a new model on R_3, involving an $su(2)$ valued scalar field, a symmetry breaking potential and up to sextic kinetic terms. We find topologically stable finite action solutions with arbitrary winding number n.

Topologically stable solitons with arbitrary winding number in R_3 are expected to exist[1], and would be interesting in two main contexts. The first, as the static solitons of a theory in $(3+1)-$dimnensional Minkowski space, an example of which is the Skyrmion[2]. The second is, as the instantons of the Euclidennised version of a theory in $(2+1)-$dimensional Minkowski space[3].

The reason that the Skyrme model is not ideally suited to this purpose is, that there is no possibility of finding an explicit expression for the Skyrmion, and at any rate even to go beyond the unite topological charge (baryon number) is not a simple task there[4].

In this note, we propose a nonlinear model in R_3, which has localised, topologically stable solutions with winding number n. Moreover, for a particular choice of the parameters characterising this model, a special submodel is found which admits (only) minimal energy solutions. These solutions are found by explicit integration of the relevant Bogomolny'i equation.

Our model is a natural generalisation of a similar one we constructed[8] in R_2. The dynamical quantity is a 2×2 Hermitian field $\Phi(\vec{x})$, which may or maynot be traceless.

Like the Skyrme model, and via the usual scaling arguments, the kinetic terms of our model must include "stabilising" terms that are at least quartic. Thus in addition to the quadratic kinetic term, we shall include both quartic and sextic[5] terms. Unlike the Skyrme model however, our model will be endowed with a symmetry-breaking potential exhibiting a dimensional parameter η, which sets the scale for the localisation of the soliton.

In its most general form, the lagrangian for our model is

$$\mathcal{L} = V(\eta,\Phi) - tr[f_1\Phi_i^2 + f_2\Phi_{ij}^2 + f_3\Phi_{ijk}^2], \tag{1}$$

where

$$\Phi_i = \partial_i\Phi, \Phi_{ij} = [\partial_i\Phi, \partial_j\Phi], \Phi_{ijk} = \{\Phi_{[ij}, \Phi_{k]}\}. \tag{2a, b, c}$$

In (2c), $[ijk]$ implies cyclic permutation, while $\{\cdot,\cdot\}$ denotes anticommutation. Both Φ_{ij} and Φ_{ijk} are by construction totally antisymmetric in their indices, which guarantees that only quadratic terms in any given component of the "velocity" Φ_i will feature in (1).

The solutions we shall seek are those that satisfy the vacumn asymptotic conditions

$$tr\ \Phi^2 \underset{r\to\infty}{\longrightarrow} \eta^2, \tag{3}$$

by virtue of the symmetry-breaking potential $V(\eta^2, \Phi^2)$ in (1). The coefficients f_1, f_2 and f_3 are in general functions of η^2 and Φ^2, and can be set equal to zero as long as this respects the criterion of topological stability. In fact, the choices of f_1, f_2, f_3 and V are controlled by this criterion.

The criteria of topological stability can most simply be illustrated for a special choice of f_1, f_2, f_3 and V, by means of the following inequalities

$$tr[\Phi_{ij} - \tfrac{1}{2}\epsilon_{ijk}(S\Phi_k + \Phi_k S)]^2 \geq 0, \tag{4a}$$

$$tr[(S\Phi_{ij} + \Phi_{[i}S\Phi_{j]} + \Phi_{ij}S) - \tfrac{1}{3}\epsilon_{ijk}(S^2\Phi_k + S\Phi_k S + \Phi_k S^2)]^2 \geq 0, \tag{4b}$$

$$tr[\Phi_{ijk} - \frac{1}{3!}\epsilon_{ijk}S^3]^2 \geq 0, \tag{4c}$$

where $S = \eta^2 - \Phi^2$. First we notice that the cross-term in each one of (4a,b,c) can easily be verified to be a total divergence. Then transferring these cross-terms to the right-hand-sides of (4a,b,c) and integrating, we would express the latter as surface integrals, which by virtue of (3) would yield the topological changes presenting lower bounds to the (action) integrals on the left-hand-side. The Lagrangians of the models can then be selected as *any linear combinations* of the positive-definite terms on the left-hand-sides of (4a,b,c).

While the inequalities (4a,b,c) guarantee the existence of topologically stable solutions, it would not be exprected to find explicit solutions in most of these models. In this respect, we would be unable to improve the situation as compared to the Skyrme model.

At this point, we recall that the reason one cannot find explicit solutions to the Skyrme model is, that the Bogomlny'i equations admit only the trivial solutions. It is exactly in this respect that our models differ from the Skyrme case.

In our case, minimal action solutions can be obtained by saturating any one of the inequalities (4a,b,c), separately. These would be solutions of the Bogomolny'i equations obtained by simply equating the expressions inside the square brackets in $4(a, b,$ and $c)$ to zero, separately. These equations however would be overdetermined in general, since they amount to three 2×2 matrix equations arising from $(4a)$, three from $(4b)$ and one from $(4c)$, involving only one 2×2 matrix field Φ.

The only subsystem for which we have a viable Bogomolny'i equation that is not overdetermined is the one arising from $(4c)$, which corresponds to $f_1 = f_2 = 0$, and $f_3 = 1$. Also here $V = tr\ S^6$, but this last specification can be relaxed, subject to ensuring V satisfies symmetry-breaking properties, and that $2\mathcal{U}$, replacing S^3 in $V = 4\,tr\,\mathcal{U}^2$, be a polynomial in Φ^2.

This specifies the promised submodel of (1) as

$$\mathcal{L}_0 = tr\ (4\mathcal{U}^2 - \frac{1}{3!}\Phi_{ijk}^2), \tag{5}$$

whose Euler-Lagrange equations are solved by the Bogomolny'i equation

$$2\mathcal{U} = \pm\frac{i}{3!}\epsilon_{ijk}\Phi_{ijk}. \tag{6}$$

The topological charge of this solution, as described above, is given by

$$q = \pm\frac{2i}{3}\int d^3x\epsilon_{ijk}\mathcal{U}\Phi_{ijk}, \tag{7}$$

which, for $\mathcal{U}$ a polynomial in Φ, can always be shown to be a surface integral, whose value would be nonzero by virtue of the symmetry-breaking nature of $V(\eta^2, \Phi^2)$ and the asymptotic condition (3). For example, for $\mathcal{U} = \frac{1}{\sigma}(\eta^2 - \Phi^2)$, the integrand of (7) $\vec{\nabla}\cdot\vec{\Omega} \approx \vec{\nabla}\cdot tr\,(2\Phi^3\vec{\nabla}\Phi \times \vec{\nabla}\Phi + \Phi\vec{\nabla}\Phi \times \Phi^2\vec{\nabla}\Phi)$.

To integrate (6) let us first re-express Φ as

$$\Phi = \phi_a\sigma_a, \quad a = 1, 2, 3, \tag{8}$$

where σ_a are the Pauli spin matrices, and we have omitted to let Φ have a nonzero trace in anticipation of this last vanishing due to the constraints of (6). Clearly U now is a polynomial in $\vec{\phi}^{\,2} = \phi_a\phi_a$ and is proportional to the unit matrix, and (6) can be reexpressed as

$$U(\eta^2, |\vec{\Phi}|^2) = \pm\epsilon_{ijk}\partial_i\phi_a\partial_j\phi_b\partial_k\phi_c\epsilon_{abc} \tag{9}$$

$$= \pm\left\|\frac{\partial\phi_a}{\partial x^i}\right\|, \tag{9'}$$

where the right-hand-side of (9), denoted by (9′), is the Jacobian for the transformation from the coordinates x_i to the coordinates ϕ_a. Accordingly, (9) can immediately be integrated formally,

$$\int d^3x = \pm\int \mathcal{U}^{-1}(\eta^2,\phi^2)d^3\phi \tag{10}$$

$$= \pm\int \mathcal{U}^{-1}(\eta^2,\phi^2)\phi^2 d\phi d\Omega, \tag{10'}$$

where $d\Omega$ in (10′) is the angular volume in terms of Θ, the polar angle and Ψ the azymuthal angle parametrising ϕ_a,

$$\phi_1 = \phi\sin\Theta\cos\Psi$$

$$\phi_2 = \phi\sin\Theta\sin\Psi \tag{11}$$

$$\phi_3 = \phi\cos\Theta.$$

Finally, we make the winding number n Antsatz

$$\Theta = \theta, \quad \Psi = n\varphi, \tag{12}$$

where (θ,φ) are the polar and azymuthal angles of $\mathbf{R}_3$ and n is an integer. Equation (10) then reduces to

$$n\int\frac{\phi^2 d\phi}{\mathcal{U}(\eta^2,\phi^2)} = \frac{1}{3}r^3 + const, \tag{13}$$

which can be integrated exactly, once $\mathcal{U}$ is specified. The integral (12) is the promised exact solution of the model described by the Lagrangian (5). The integration constant in (13) is to be chosen subject to the condition that

$$\phi(r)\underset{r\to 0}{\longrightarrow} 0, \tag{14}$$

which is a necessary condition for the current $\vec{\Omega}$ in the integrand of (7), namely $\vec{\nabla}\cdot\vec{\Omega}$, to be well- defined[6].

That the field Φ specified by the Ansatz (8), (11) and (12) is a minimal action solution of the model (5) with winding number n, can be verified before integrating the left-hand-side of (13), *provided* that we assume the result of this integration respects the two boundary conditions (3) and (14).

The result of substituting the values of Φ from (8), (11) and (12) into (7) is

$$q = \pm 2^3\int \mathcal{U}(\eta^2,\phi^2)\phi^2 d\phi\sin\Theta d\Theta d\Psi = \pm 4\pi n\, I, \tag{15}$$

where the integral I equals $2^3\int \mathcal{U}(\eta^2,\phi^2)\phi^2 d\phi$. By virtue of the selfduality equation (6), q is equal to the action integral, which is therefore given by the winding number n.

To illustrate our result with a concrete example, again we consider the simplest non-trivial form $\mathcal{U} = \frac{1}{\sigma}(\eta^2 - \Phi^2)$. Integrating (13) explicitly then yields

$$r^3 = 3n\sigma\left[-\phi + \tfrac{1}{2}\eta\ln\left(\frac{\eta+\phi}{\eta-\phi}\right)\right] \tag{16}$$

which clearly respects both boundary conditions (3) and (14). The integral I normalising the topological charge (15) is then found to be

$$I = \frac{8}{\sigma}\left[\frac{1}{3}\eta^2\phi^3 - \frac{1}{5}\phi^5\right]_{\phi(0)}^{\phi(\infty)}, \tag{17}$$

which, due to $\phi(0) = 0$ and $\phi(\infty) = \eta$, is a finite non-vanishing inequality.

This completes our discussion of the classical properties of our models (1), and the special case (5). What remains is to study the quantum mechanical properties. In this respect, here we shall suffice to note that both (1) and (5) possess an additional invariance under

$$\Phi \to A\Phi A^{-1}, \tag{18}$$

where A is an arbitrary $su(2)$ matrix. This means that if we interpret our classical lumps as *static* solitons in $(3+1)$−dimensional space, we can treat A as a collective coordinate

$A = A(t)$ as was done in ref. [7]. Our models are therefore alternatives to the Skyrme model, except that some of them have explicit integrals and an infinite tower of excited states. In addition, we hope that our models are of intrinsic interest as instantons in R_3 within the context of $(2+1)-$dimensional physics[3].

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